Initial Boundary Value Problems for Integrable Nonlinear Equations: a Riemann–Hilbert Approach

Dmitry Shepelsky

Institute for Low Temperature Physics, Kharkiv, Ukraine

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IBVP for focusing NLS

with decaying initial data and (asymptotically) periodic boundary conditions

Let $q(x,t)$ be the solution of the IBV problem for focusing NLS:

- $\mathrm{i}q_t + q_{xx} + 2|q|^2 q = 0, \qquad x > 0, t > 0,$
- $q(x,0) = q_0(x)$ fast decaying as $x \to +\infty$
- $q(0,t) = g_0(t)$ time-periodic $\boxed{g_0(t) = \alpha\,\mathrm{e}^{2\mathrm{i}\omega t}}$ $\quad \alpha > 0, \omega \in \mathbb{R}$
  $(q(0,t) - \alpha\,\mathrm{e}^{2\mathrm{i}\omega t} \to 0$ as $t \to +\infty)$

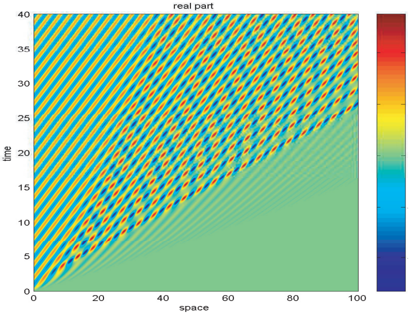
▷ Question: How behaves $q(x,t)$ for large t?

▷ Numerics: Qualitatively different pictures for parameter ranges:

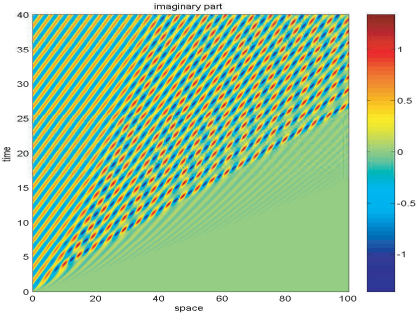
(i) $\omega < -3\alpha^2$

(ii) $-3\alpha^2 < \omega < \frac{\alpha^2}{2}$

(iii) $\omega > \frac{\alpha^2}{2}$



Real part $\mathrm{Re}\, q(x,t)$



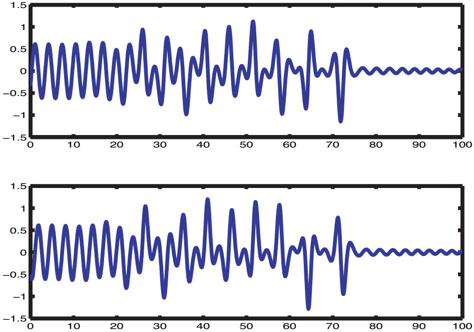
Imaginary part $\mathrm{Im}\, q(x,t)$

$\alpha = \sqrt{3/8}$, $\omega = -13/8$

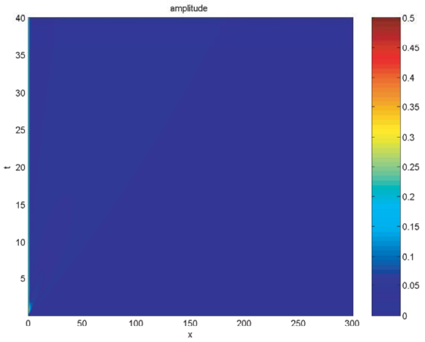
$q_0(x) \equiv 0$, $g_0(t) = \alpha e^{2i\omega t} + \mathrm{O}(e^{-10t^2})$

Numerics for $\omega < -3\alpha^2$, II

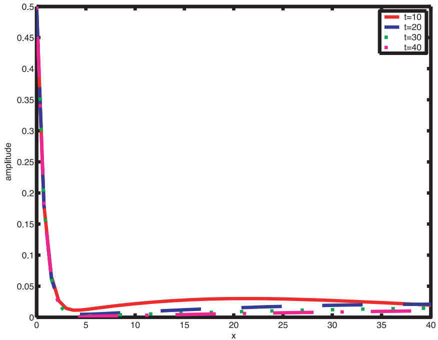
Numerical solution for $t = 20$, $0 < x < 100$.
Upper: real part $\mathrm{Re}\, q(x, 20)$. Lower: imaginary part $\mathrm{Im}\, q(x, 20)$.



Numerics for $\omega \geq \alpha^2/2$



Amplitude of $q(x,t)$



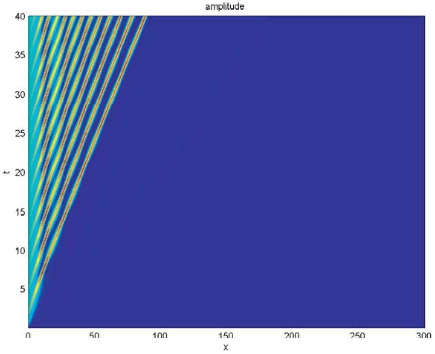
Amplitude for $t = 10, \ldots$

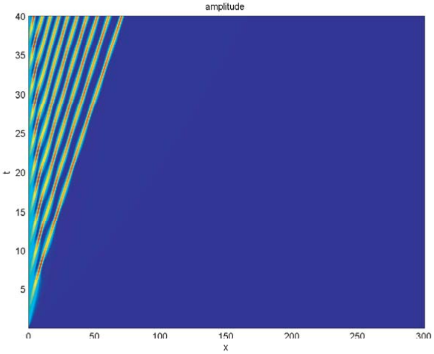
$\alpha = 0.5,\ \omega = 1,\ \omega \geq \alpha^2/2,$

$q_0(x) \equiv 0,\ g_0(t) = \alpha e^{2i\omega t} + O(e^{-10t^2})$

Numerics for $-3\alpha^2 < \omega < \alpha^2/2$

Amplitude of $q(x,t)$





$\alpha = 0.5$
$\omega = -2\alpha^2 = -0.5$

$\alpha = 0.5$
$\omega = -\alpha^2 = -0.25$

$$q_0(x) \equiv 0, \qquad g_0(t) = \alpha\, e^{2i\omega t} + O(e^{-10t^2})$$

Inverse Scattering Transform for whole line problems, I

A nonlinear PDE in dimension 1+1 $q_t = F(q, q_x, ...)$ integrable $\Leftrightarrow$ it is compatibility condition for 2 linear equations (Lax pair): matrix-valued (2×2); involve parameter k

- $\Psi_x = U\Psi, \quad \Psi_t = V\Psi$

$$U = U(q; k), \quad V = V(q, q_x, \ldots; k)$$

- $q_t = F(q, q_x, ...) \Longleftrightarrow \Psi_{xt} = \Psi_{tx}$ for all k: $\quad U_t - V_x = [V, U]$

Cauchy (whole line) problem: given $q(x, 0) = q_0(x)$, $x \in (-\infty, \infty)$ ($q_0(x) \to 0$ as $|x| \to \infty$), find $q(x, t)$.

Solution: $q(x, 0) \to s(k; 0) \to s(k; t) \to q(x, t)$.

- $q(x, 0) \to s(k; 0)$: direct spectral (scattering) problem for x-equation of the Lax pair
- $s(k; 0) \to s(k; t)$: evolution of spectral functions (linear!)
- $s(k; t) \to q(x, t)$: inverse spectral (scattering) problem for x-equation

In the case of NLS:

- $U = -\mathrm{i}k\sigma_3 + Q; \quad V = 2\mathrm{i}k^2\sigma_3 + \tilde{Q} \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$
  with $Q = \begin{pmatrix} 0 & q \\ -\bar{q} & 0 \end{pmatrix}$, $\tilde{Q} = 2kQ - \mathrm{i}Q_x\sigma_3 - \mathrm{i}|q|^2\sigma_3$
- direct scattering: introduce $\Psi_\pm$ dedicated solutions of $\Psi_x = U(q(x,t);k)\Psi$:
  $$\Psi_\pm \sim \Psi_0 (= \mathrm{e}^{-\mathrm{i}kx\sigma_3}), x \to \pm\infty$$
  Then $\Psi_+(x;t,k) = \Psi_-(x;t,k)s(k;t)$ (scattering relation)
- evolution of scattering functions:
  $s_t = 2ik^2[\sigma_3, s] \Rightarrow s(k;t) = \mathrm{e}^{-\mathrm{i}2k^2t\sigma_3}s(k;0)\mathrm{e}^{\mathrm{i}2k^2t\sigma_3}$

- $s(k;t) \to q(x,t)$: inverse spectral (scattering) problem for x-equ. Can be done in terms of Riemann-Hilbert problem (RHP):

  find M: 2×2, piecewise analytic in $\mathbb{C}$ (w.r.t. k) s.t.

  - $M_+(x,t;k) = M_-(x,t;k)\mathrm{e}^{-\mathrm{i}(2k^2t+kx)\sigma_3}\tilde{s}(k;0)\mathrm{e}^{\mathrm{i}(2k^2t+kx)\sigma_3}, \quad k \in \mathbb{R}$

  $$(s(k;0) \to \tilde{s}(k;0): \quad \text{algebraic manipulations})$$

  - $M \to I$ as $|k| \to \infty$
  - in case of M piecewise meromorphic: residue conditions at poles

  Then $q(x,t) = 2\mathrm{i}\lim_{k\to\infty}(kM_{12}(x,t,k))$.

Hint: M is constructed from columns of Ψ_+ and Ψ_- following their analyticity properties w.r.t k; then jump relation for RHP is a re-written scattering relation for $\Psi_\pm$.

Thus the Inverse Scattering Transform (IST) method: a kind of change of variables that linearizes the problem.

Importance: most efficient for studying long-time behavior of solutions of Cauchy problem with general initial data. This is due to explicit (x, t)-dependence of data for the RHP (jump matrix; residue conds. if any), which makes possible to apply a nonlinear version of the steepest descent method (Deift, Zhou, 2993) for studying asymptotic behavior of solutions of relevant Riemann–Hilbert problems with oscillatory jump conditions (linear analogue: asymptotic evaluation of contour integrals by Laplace or stationary phase methods).

General scheme for boundary value problems via IST

Natural problem: to adapt (generalize) the RHP approach to boundary-value (initial-boundary value) problems for integrable equations.

Data for an IBV problem (e.g, in domain $x > 0$, $t > 0$):

(i) Initial data: $q(x,0) = q_0(x)$, $x > 0$

(ii) Boundary data: $q(0,t) = g_0(t)$, $q_x(0,t) = g_1(t), \ldots$.

Question: How many boundary values?

For a well-posed problem: roughly “half” the number of x-derivatives.

For NLS: one b.c. (e.g., $q(0,t) = g_0(t)$).

General idea for IBV: use both equations of the Lax pair as spectral problems.

Common difficulty: spectral analysis of the t-equation on the boundary ($x = 0$) involves more functions (boundary values $q(0,t), q_x(0,t), \ldots$) than possible data for a well-posed problem.

Half-line problem for NLS

For NLS: t-equation involves q and q_x; hence for the (direct) spectral analysis at $x = 0$ one needs $q(0, t)$ and $q_x(0, t)$. Assume that we are given the both. Then one can define two sets of spectral functions coming from the spectral analysis of x-equation and t-equation.

(i) $q_0 \mapsto \{a(k), b(k)\}$ (direct problem for x-equ); $\quad s \equiv \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}$

$\{g_0, g_1\} \mapsto \{A(k), B(k)\}$ (direct problem for t-equ)

(ii) From the spectral functions $\{a(k), b(k), A(k), B(k)\}$, the jump matrix $J(x, t, k)$ for the Riemann-Hilbert problem is constructed: $\{a(k), b(k), A(k), B(k)\} \mapsto J_0(k)$:

$$J(x, t, k) = \mathrm{e}^{-\mathrm{i}(2k^2 t + kx)\sigma_3} J_0(k) \mathrm{e}^{\mathrm{i}(2k^2 t + kx)\sigma_3}$$

(notice the same explicit dependence on (x, t)!) The jump conditions are across a contour Γ determined by the asymptotic behavior of $g_0(t)$ and $g_1(t)$

(iii) The RHP is formulated relative to Γ:
$M_+(x, t, k) = M_-(x, t, k) J(x, t, k)$, $k \in \Gamma$; $\qquad M \to I$ as $k \to \infty$

(iv) Similarly to the Cauchy (whole-line) problem, the solution of the IBV (half-line) problem is given in terms of the solution of the RHP:
$q(x, t) = 2\mathrm{i} \lim_{k \to \infty} (k M_{12}(x, t, k))$

Direct spectral problems for NLS in half-strip $x > 0$, $0 < t < T$

- Given $q_0(x)$, determine $a(k), b(k)$: $a(k) = \Phi_2(0,k)$, $b(k) = \Psi_1(0,k)$, where vector $\Phi(x,k)$ is the solution of the x-equation evaluated at $t = 0$:

$$\Phi_x + \mathrm{i}k\sigma_3\Phi = Q(x,0,k)\Phi, \quad 0 < x < \infty, \mathrm{Im}\, k \geq 0$$

$$\Phi(x,k) = e^{\mathrm{i}kx}\left(\begin{pmatrix}0\\1\end{pmatrix} + o(1)\right) \text{ as } x \to \infty,$$

$$Q(x,0,k) = \begin{pmatrix} 0 & q_0(x) \\ -\bar{q}_0(x) & 0 \end{pmatrix}$$

- Given $\{g_0(t), g_1(t)\}$, determine $A(k;T), B(k;T)$: $A(k;T) = \mathrm{e}^{2\mathrm{i}k^2T}\overline{\tilde{\Phi}_2(T,\bar{k})}$, $B(k;T) = -\mathrm{e}^{2\mathrm{i}k^2T}\tilde{\Phi}_2(T,k)$, where vector $\tilde{\Phi}(x,k)$ is the solution of the t-equation evaluated at $x = 0$:

$$\tilde{\Phi}_t + 2\mathrm{i}k^2\sigma_3\tilde{\Phi} = \tilde{Q}(0,t,k)\tilde{\Phi}, \quad 0 < t < T,$$

$$\tilde{\Phi}(0,k) = \begin{pmatrix}0\\1\end{pmatrix}$$

$$\tilde{Q}(0,t,k) = \begin{pmatrix} -|g_0(t)|^2 & 2kg_0(t) - \mathrm{i}g_1(t) \\ 2k\bar{g}_0(t) + \mathrm{i}\bar{g}_1(t) & |g_0(t)|^2 \end{pmatrix}$$

RHP for NLS in half-strip $x > 0$, $0 < t < T$

- Contour: $\Gamma = \mathbb{R} \cup \mathrm{i}\mathbb{R}$
- Jump matrix:

$$J_0(k) = \begin{cases} \begin{pmatrix} 1+|r(k)|^2 & \bar{r}(k) \\ r(k) & 1 \end{pmatrix}, & k > 0, \\ \begin{pmatrix} 1 & 0 \\ C(k;T) & 1 \end{pmatrix}, & k \in \mathrm{i}\mathbb{R}_+, \\ \begin{pmatrix} 1 & \bar{C}(\bar{k};T) \\ 0 & 1 \end{pmatrix}, & k \in \mathrm{i}\mathbb{R}_-, \\ \begin{pmatrix} 1+|r(k)+C(k;T)|^2 & \bar{r}(k)+\bar{C}(k;T) \\ r(k)+C(k;T) & 1 \end{pmatrix}, & k < 0, \end{cases}$$

where $r(k) = \dfrac{\bar{b}(k)}{a(k)}$, $C(k;T) = -\dfrac{\overline{B(\bar{k};T)}}{a(k)d(k;T)}$ with $d = a\bar{A} + b\bar{B}$

(also works for $T = +\infty$ if $g_0(t), g_1(t) \to 0$, $t \to \infty$)

Eigenfunctions for NLS in half-strip $x > 0, 0 < t < T$

Hint: Define $\Psi_j(x,t,k)$, $j = 1, 2, 3$ solutions (2×2) of the Lax pair equations normalized at "corners" of the (x,t)-domain where the IBV problem is formulated:

1. $\Psi_1(0,T,k) = \mathrm{e}^{-2\mathrm{i}k^2 T\sigma_3}$ ($\Psi_1(0,t,k) \simeq \mathrm{e}^{-2\mathrm{i}k^2 t\sigma_3}$ as $t \to \infty$)
2. $\Psi_2(0,0,k) = I$
3. $\Psi_3(x,0,k) \simeq \mathrm{e}^{-\mathrm{i}kx\sigma_3}$ as $x \to \infty$

They can be constructed as solutions of integral equations let $\mu_j = \Psi_j \mathrm{e}^{(\mathrm{i}kx+2\mathrm{i}k^2 t)\sigma_3}$; then

$$\mu_1(x,t,k) = I + \int_0^x \mathrm{e}^{\mathrm{i}k(x-y)\hat{\sigma}_3}(Q\mu_1)(y,t,k)\,\mathrm{d}y - \mathrm{e}^{\mathrm{i}kx\hat{\sigma}_3}\int_t^T \mathrm{e}^{-4\mathrm{i}k^3(t-\tau)\hat{\sigma}_3}(\tilde{Q}\mu_1)(0,\tau,k)\,\mathrm{d}\tau, \tag{1}$$

$$\mu_2(x,t,k) = I + \int_0^x \mathrm{e}^{\mathrm{i}k(x-y)\hat{\sigma}_3}(Q\mu_2)(y,t,k)\,\mathrm{d}y + \mathrm{e}^{\mathrm{i}kx\hat{\sigma}_3}\int_0^t \mathrm{e}^{-4\mathrm{i}k^3(t-\tau)\hat{\sigma}_3}(\tilde{Q}\mu_2)(0,\tau,k)\,\mathrm{d}\tau, \tag{2}$$

$$\mu_3(x,t,k) = I - \int_x^\infty \mathrm{e}^{\mathrm{i}k(x-y)\hat{\sigma}_3}(Q\mu_3)(y,t,k)\,\mathrm{d}y. \tag{3}$$

Here $\mathrm{e}^{\hat{A}}B := \mathrm{e}^{A}B\mathrm{e}^{-A}$.

Scattering for x-and t-equations

Integral equations: convenient for studying properties w.r.t k: analyticity; boundedness. Indeed, this follows the analyticity/boundedness of the involved exponentials.

Being simultaneous solutions of x-and t-equation, they are related by two scattering relations:

(i) $\Psi_3(x,t,k) = \Psi_2(x,t,k)s(k) \qquad s = \begin{pmatrix} \bar{a} & b \\ -\bar{b} & a \end{pmatrix}$

(ii) $\Psi_1(x,t,k) = \Psi_2(x,t,k)S(k;T) \qquad S = \begin{pmatrix} \bar{A} & B \\ -\bar{B} & A \end{pmatrix}$

Then M is constructed from columns of Ψ_1, Ψ_2 and Ψ_3 following their analyticity and boundedness properties w.r.t k, and the jump relation for RHP is re-written scattering relations (i)+(ii) for Ψ_j.

For NLS in half-strip ($T < \infty$) or in quarter plane ($T = \infty$) with $g_j(t) \to 0$ as $t \to \infty$: first column of $\Psi_1(x,t,k)\mathrm{e}^{(-\mathrm{i}kx - 2\mathrm{i}k^2 t)\sigma_3}$ is bounded in $\{k : \mathrm{Im}\, k \geq 0, \mathrm{Im}\, k^2 \leq 0\}$, etc., which leads to $\Gamma = \mathbb{R} \cup \mathrm{i}\mathbb{R}$.

Compatibility of boundary values: Global Relation

The fact that the set of initial and boundary values $\{q_0(x), g_0(t), g_1(t)\}$ cannot be prescribed arbitrarily (as data for IBVP) must be reflected in spectral terms.

Indeed, from scattering relations (i)+(ii): $S^{-1}(k;T)s(k) = \Psi^{-1}(x,t,k)\Psi_3(x,t,k)$. Evaluating this at $x = 0$, $t = T$ and using analyticity and boundedness properties of Ψ_j one deduce for the (12) entry of $S^{-1}s$:

$$\boxed{A(k;T)b(k) - a(k)B(k;T) = O\left(\frac{e^{4ik^2T}}{k}\right), \quad k \to \infty, \quad \mathrm{Im}\, k \geq 0, \mathrm{Re}\, k \geq 0}$$

This relation is called Global Relation (GR): it characterizes the compatibility of $\{q_0(x), g_0(t), g_1(t)\}$ in spectral terms.

Typical theorem: Consider the IBVP with given $q_0(x)$ and $g_0(t)$. Assume that there exists $g_1(t)$ such that the associated spectral functions $\{a(k), b(k), A(k), B(k)\}$ satisfy the Global Relation. Then the solution of the IBVP is given in terms of the solution of the RHP above. Moreover, it satisfies also the b.c. $q_x(0,t) = g_1(t)$.

Using Global Relation

1. GR can be used to describe the Dirichlet-to-Neumann map, i.e., to derive $g_1(t) = q_x(0,t)$ from $\{q_0(x) = q(x,0),\ g_0(t) = q(0,t)\}$;
$$g_1(t) = \frac{g_0(t)}{\pi} \int_{\partial D} e^{-2ik^2 t} \left(\tilde{\Phi}_2(t,k) - \tilde{\Phi}_2(t,-k)\right) dk + \frac{4i}{\pi} \int_{\partial D} e^{-2ik^2 t} k r(k) \overline{\tilde{\Phi}_2(t,\bar{k})} dk$$
$$+ \frac{2i}{\pi} \int_{\partial D} e^{-2ik^2 t} (k[\tilde{\Phi}_1(t,k) - \tilde{\Phi}_1(t,-k)] + i g_0(t)) dk$$
But: nonlinear! (g_1 is involved in the construction of $\tilde{\Phi}_j$)
   - In the small-amplitude limit, reduces to a formula giving $g_1(t)$ in terms of $g_0(t)$ and $q_0(x)$ (via $r(k)$); here NLS reduces to a linear equation $iq_t + q_{xx} = 0$.
   - This suggests perturbative approach: given $g_0(t)$ say periodic with small amplitude, one derives a perturbation series for $g_1(t)$ with periodic terms.
2. For some particular b.c. (called linearizable): use additional k-invariance in t-equation for expressing all ingredients in jump matrix in terms of spectral data associated with initial data only. Example: IBVP with homogeneous Dirichlet b.c. ($g_0(t) \equiv 0$); also Neumann b.c. ($g_1(t) \equiv 0$) and mixed (Robin) b.c.
3. For $T = \infty$: if $g_0(t) \to 0$ as $t \to \infty$ and assuming that $g_1(t) \to 0$, the GR takes the form
$$A(k)b(k) - a(k)B(k) = 0, \qquad \operatorname{Im} k \geq 0, \operatorname{Re} k \geq 0$$
Since the structure of the RHP is similar to that for whole-line problem, one can study long-time behavior of solution via nonlinear steepest descent.
But: parameters of the asymptotics - in terms of $A(k)$, $B(k)$, which are not known for a well-posed IBVP.

IBV problem with oscillatory b.c.

For $T = \infty$: the approach can be implemented for boundary values non-decaying as $t \to \infty$. But for this: one needs correct large-time behavior of $g_1(t)$ associated with that of the given $g_0(t)$; this is because both $g_0(t)$ and $g_1(t)$ determine the spectral problem for t-equation and thus the structure of associated spectral functions $A(k)$, $B(k)$.

Dirichlet-to-Neumann map

Let $q(0,t)=\alpha\,\mathrm{e}^{2\mathrm{i}\omega t}$ $(q(0,t)-\alpha\,\mathrm{e}^{2\mathrm{i}\omega t}\to 0,\ t\to\infty)$
Neumann values ($q_x(0,t)$):

(i) numerics:

$$q_x(0,t)\simeq c\,\mathrm{e}^{2\mathrm{i}\omega t} \qquad c=\begin{cases}2\mathrm{i}\alpha\sqrt{\frac{\alpha^2-\omega}{2}}, & \omega\le -3\alpha^2\\ \pm\alpha\sqrt{2\omega-\alpha^2}, & \omega\ge\frac{\alpha^2}{2}\end{cases}$$

(ii) theoretical results: agreed with numerics (for all $x>0$, $t>0$) provided c as above.

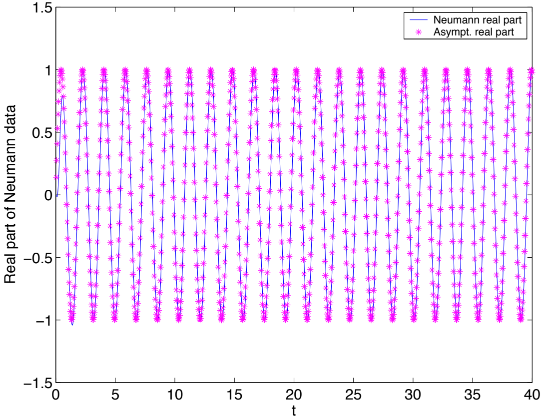
Question: Why these particular values of c?

(the spectral mapping $\{g_0,g_1\}\mapsto\{A(k),B(k)\}$ is well-defined for any $c\in\mathbb{C}$!)

Idea: Use the global relation (its impact on analytic properties of $A(k)$, $B(k)$) to specify admissible values of parameters α,ω,c.

Numerics: Neumann values, $\omega < -3\alpha^2$

Neumann values $q_x(0, t)$ for $\alpha = 0.5$ and $\omega = -1.75$.



The numerics agree with $q_x(0, t) = 2\mathrm{i}\alpha\beta\, q(0, t)$.

Theorem 1: $\omega < -3\alpha^2$

Consider the Dirichlet initial-boundary value problem for NLS$_+$

- $\mathrm{i}q_t + q_{xx} + 2|q|^2 q = 0, \quad x, t \in \mathbb{R}_+,$
- $q(x,0) = q_0(x)$ fast decaying,
- $q(0,t) = g_0(t) \equiv \alpha\, \mathrm{e}^{2\mathrm{i}\omega t}$ time-periodic, $\alpha > 0$, $\boxed{\omega < -3\alpha^2}$
- $q_0(0) = g_0(0)$.

▷ Assume $q_x(0,t) \sim 2\mathrm{i}\alpha\beta\, \mathrm{e}^{2\mathrm{i}\omega t}$ as $t \to +\infty$ with $\beta = \sqrt{\frac{\alpha^2-\omega}{2}}$.

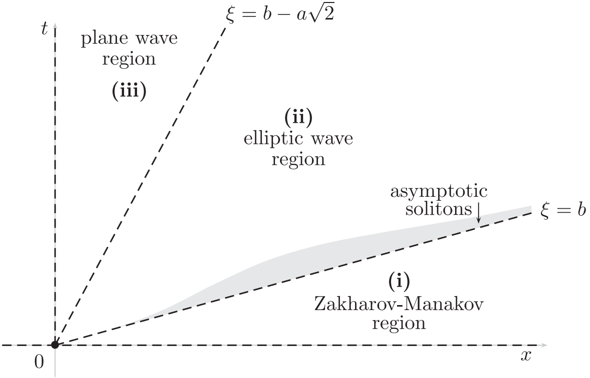
Let $\xi := \frac{x}{4t}$. Then for large t, the solution $q(x,t)$ behaves differently in 3 sectors of the (x,t)-quarter plane (in agreement with numerics):

(i) $\xi > \beta \implies q(x,t)$ looks like decaying modulated oscillations of Zakharov-Manakov type.

(ii) $\sqrt{\beta^2 - 2\alpha^2} < \xi < \beta \implies q(x,t)$ looks like a modulated elliptic wave.

(iii) $0 \leq \xi < \sqrt{\beta^2 - 2\alpha^2} \implies q(x,t)$ looks like a plane wave.

Three regions for $\omega < -3\alpha^2$



Regions in the (x, t)-quarter-plane: $\xi = \frac{x}{4t}$, $\beta = \sqrt{\frac{\alpha^2 - \omega}{2}}$

Asymptotics for $\omega < -3\alpha^2$

- $\xi = \frac{x}{4t} > \beta$:

$$q(x,t) = \frac{1}{\sqrt{t}}\,\rho(-\xi)\mathrm{e}^{4\mathrm{i}\xi^2 t + 2\mathrm{i}\rho^2(-\xi)\log t + \mathrm{i}\phi(-\xi)} + \mathrm{o}\Big(\frac{1}{\sqrt{t}}\Big)$$

- $\beta - \alpha\sqrt{2} < \xi < \beta$:

$$q(x,t) \simeq [\alpha + \mathrm{Im}\, d(\xi)]\frac{\theta_3[B_g t/2\pi + B_\omega \Delta/2\pi + U_-]}{\theta_3[B_g t/2\pi + B_\omega \Delta/2\pi + U_+]}\,\frac{\theta_3[U_+]}{\theta_3[U_-]}\mathrm{e}^{2\mathrm{i}g_\infty(\xi)t - 2\mathrm{i}\phi(\xi)}$$

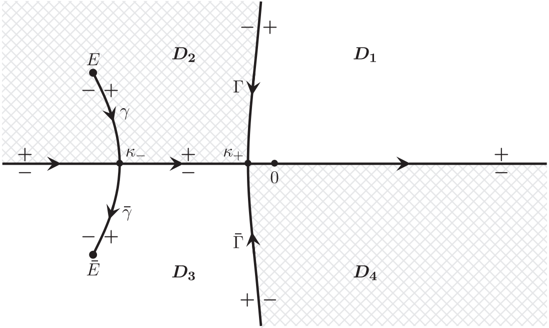
- $0 < \xi < \beta - \alpha\sqrt{2}$:

$$q(x,t) = \alpha\,\mathrm{e}^{2\mathrm{i}[\beta x + \omega t - \phi(\xi)]} + \mathrm{O}\Big(\frac{1}{\sqrt{t}}\Big)$$

The parameters (functions of ξ) $d, B_g, B_\omega, g_\infty, \phi$ can be expressed in terms of the spectral functions associated to IB data $\{q_0(x), g_0(t), g_1(t)\}$.

The RHP for NLS: the contour

for $\omega < -3\alpha^2$, assuming $q_x(0,t) \sim 2\mathrm{i}\alpha\beta\,\mathrm{e}^{2\mathrm{i}\omega t}$



$\hat\Gamma := \mathbb{R} \cup \gamma \cup \bar\gamma \cup \Gamma \cup \bar\Gamma$ with $E = -\beta + \mathrm{i}\alpha$.

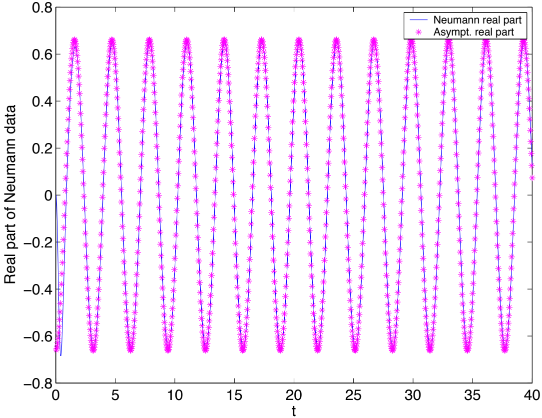
The RHP for NLS: the jump matrix

$$J(x,t;k) = \begin{cases} \begin{pmatrix} 1 & -\bar{\rho}(k)\mathrm{e}^{-2\mathrm{i}t\theta(k)} \\ -\rho(k)\mathrm{e}^{2\mathrm{i}t\theta(k)} & 1+|\rho(k)|^2 \end{pmatrix} & k \in (-\infty, \kappa_+), \\ \begin{pmatrix} 1 & -\bar{r}(k)\mathrm{e}^{-2\mathrm{i}t\theta(k)} \\ -r(k)\mathrm{e}^{2\mathrm{i}t\theta(k)} & 1+|r(k)|^2 \end{pmatrix} & k \in (\kappa_+, \infty), \\ \begin{pmatrix} 1 & 0 \\ c(k)\mathrm{e}^{2\mathrm{i}t\theta(k)} & 1 \end{pmatrix} & k \in \Gamma, \\ \begin{pmatrix} 1 & \bar{c}(\bar{k})\mathrm{e}^{-2\mathrm{i}t\theta(k)} \\ 0 & 1 \end{pmatrix} & k \in \bar{\Gamma}, \\ \begin{pmatrix} 1 & 0 \\ f(k)\mathrm{e}^{2\mathrm{i}t\theta(k)} & 1 \end{pmatrix} & k \in \gamma, \\ \begin{pmatrix} 1 & -\bar{f}(\bar{k})\mathrm{e}^{-2\mathrm{i}t\theta(k)} \\ 0 & 1 \end{pmatrix} & k \in \bar{\gamma}. \end{cases}$$

where $\boxed{\theta(k) = \theta(k,\xi) = 2k^2 + 4\xi k}$ with $\boxed{\xi = \dfrac{x}{4t}}$

Numerics: Neumann values, $\omega \geq \alpha^2/2$

Neumann values $q_x(0,t)$ for $\alpha = 0.5$ and $\omega = 1$.



The numerics agree with $q_x(0,t) = 2\alpha\hat{\beta}\, q(0,t)$.

Theorem 2: $\omega \geq \alpha^2/2$

Consider the Dirichlet initial-boundary value problem for NLS$_+$

- $\mathrm{i}q_t + q_{xx} + 2|q|^2 q = 0, \quad x, t \in \mathbb{R}_+$.
- $q(x, 0) = q_0(x)$ fast decaying.
- $q(0, t) = g_0(t) \equiv \alpha\, \mathrm{e}^{2\mathrm{i}\omega t}$ time-periodic, $\alpha > 0$, $\boxed{\omega \geq \alpha^2/2}$
- $q_0(0) = g_0(0)$.

▷ Assume that $q_x(0, t) \sim 2\alpha\hat{\beta}\, \mathrm{e}^{2\mathrm{i}\omega t}$ with $\hat{\beta} = \pm\frac{1}{2}\sqrt{2\omega - \alpha^2}$.

Then for $\xi = \frac{x}{4t} > \varepsilon > 0$,

$$q(x, t) = \frac{1}{\sqrt{t}}\, \rho(-\xi) \mathrm{e}^{4\mathrm{i}\xi^2 t + 2\mathrm{i}\rho^2(-\xi)\log t + \mathrm{i}\phi(-\xi)} + \mathrm{o}\Big(\frac{1}{\sqrt{t}}\Big)$$

(decaying modulated oscillations of Zakharov-Manakov type), where parameters $\rho(\xi)$ and $\phi(\xi)$ are determined by the IB data $q_0(x)$, $g_0(t)$, and $g_1(t)$ via the spectral functions $a(k), b(k), A(k), B(k)$.

Theorem 3: admissible $\{\alpha, \omega, c\}$

Let $q(x,t)$ be a solution of the NLS ($x > 0, t > 0$) such that:

- $q(0,t) - \alpha\, \mathrm{e}^{2\mathrm{i}\omega t} \to 0$ as $t \to +\infty$ ($\alpha > 0, \omega \in \mathbb{R}$)
- $q_x(0,t) - c\, \mathrm{e}^{2\mathrm{i}\omega t} \to 0$ as $t \to +\infty$, for some $c \in \mathbb{C}$
- $q(x,t) \to 0$ as $x \to +\infty$ ($\forall t \geq 0$)

Then the admissible values of $\{\alpha, \omega, c\}$ are given by:

- $\omega \leq -3\alpha^2$, $c = 2\mathrm{i}\alpha\sqrt{\frac{\alpha^2 - \omega}{2}}$
- $\omega \geq \frac{\alpha^2}{2}$, $c = \pm\alpha\sqrt{2\omega - \alpha^2}$.

Idea of proof

1. For all $\{g_0, g_1\}$ whose asymptotics is associated with $\{\alpha, \omega, c\}$, where $c = c_1 + \mathrm{i}c_2$, the t-equation of the Lax pair for the NLS (at $x = 0$) has a solution $\Phi(t, k)$, $k \in \Sigma = \{k : \operatorname{Im}\Omega(k) = 0\}$, s.t.

   $\Phi(t, k) = \Psi(t, k)(1 + o(1))$ as $t \to +\infty$, where

$$\Psi(t, k) = \mathrm{e}^{\mathrm{i}\omega t\sigma_3} E(k) \mathrm{e}^{-\mathrm{i}\Omega(k)t\sigma_3},$$

$$E(k) = \sqrt{\frac{2\Omega - H}{2\Omega}} \begin{pmatrix} 1 & -\frac{\mathrm{i}H}{2ak-\mathrm{i}\bar{c}} \\ -\frac{\mathrm{i}H}{2ak+\mathrm{i}c} & 1 \end{pmatrix} \text{ with } H(k) = \Omega(k) - 2k^2 + a^2 - \omega,$$

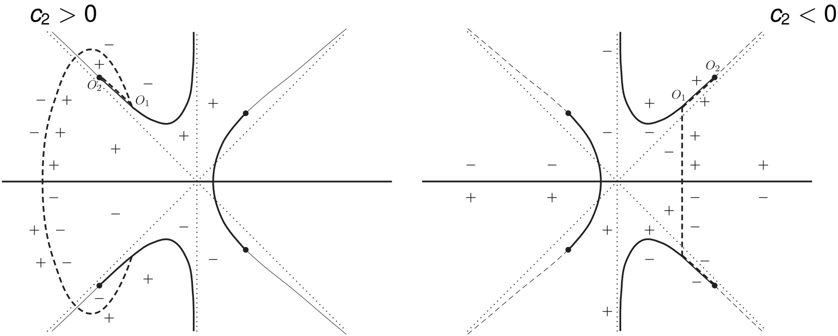
$$\Omega^2(k) = k^4 + 4\omega k^2 - 4\alpha c_2 k + (\alpha^2 - \omega)^2 + c_1^2 + c_2^2.$$

2. $\Gamma = \Sigma \cup \{\text{branch cuts}\}$ is the contour for the RH problem for the inverse spectral mapping $\{A(k), B(k)\} \to \{g_0, g_1\}$.
3. Compatibility of $\{q_0, g_0, g_1\}$ in spectral terms: global relation

$$A(k)b(k) - a(k)B(k) = 0, \quad k \in D = \{k : \operatorname{Im} k > 0, \operatorname{Im}\Omega(k) > 0\}.$$

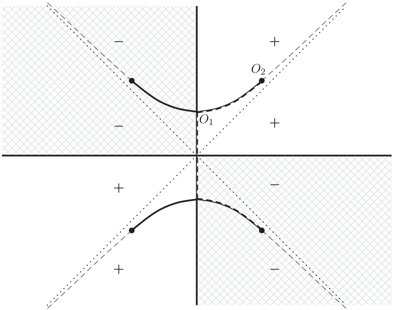
4. Existence of a (finite) arc of $\Gamma_0 = \Sigma \cap \{\text{branch cuts}\}$ in D contradicts the global relation (particularly, the continuity of $b(k)$ and $a(k)$ across the arc).

Non-admissible spectral curves: $\omega > 0$, I

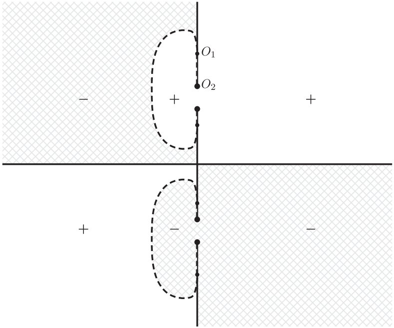


Non-admissible spectral curves: $\omega > 0$, II

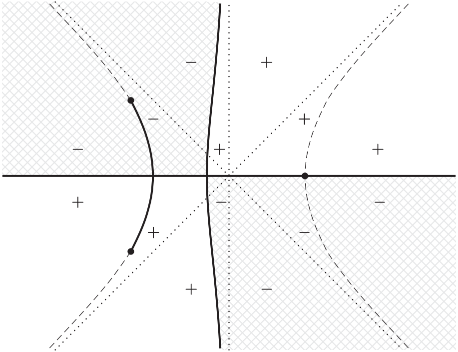
$c_2 = 0,\ 0 < \omega < \frac{\alpha^2}{2}$



$c_2 = 0,\ c_1^2 < \alpha^2(2\omega - \alpha^2)$



Admissible spectral curves: $\omega < 0$



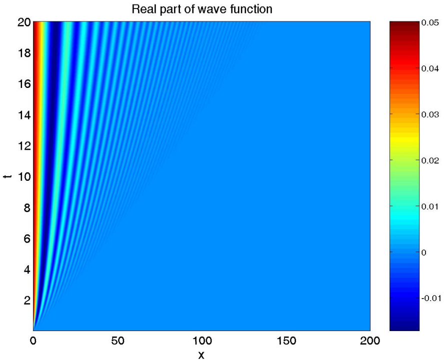
Range $\omega < 0$, $c_2 > 0$: the only admissible case is when the finite arc of $\{\mathrm{Im}\,\Omega(k) = 0\}$ lying on the right branch of the curve $\{\mathrm{Im}\,\Omega^2(k) = 0\}$ degenerates to a point on $\mathbb{R}$, i.e., when $\Omega^2(k)$ has a double, positive zero. In terms of $\{\alpha, \omega, c\}$, this corresponds to:

$$c_1 = 0,\ c_2 = \alpha\sqrt{2(\alpha^2 - \omega)}.$$

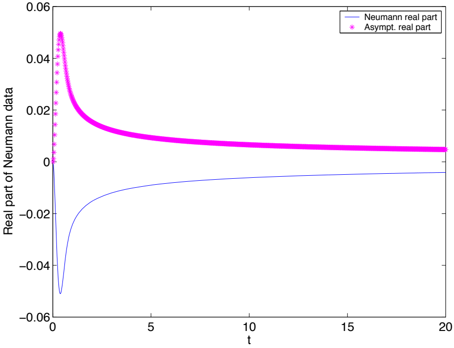
Numerics for $-3\alpha^2 < \omega < \alpha^2/2$, II

$\boxed{\alpha = 0.05, \quad \omega = 0}$

$q_0(x) \equiv 0, \;\; g_0(t) = \alpha + \mathrm{O}(\mathrm{e}^{-10t^2})$



Real part of $q(x,t)$

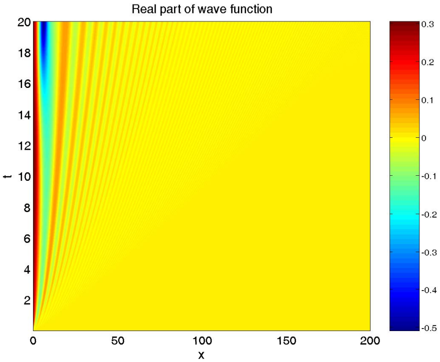


Neumann data

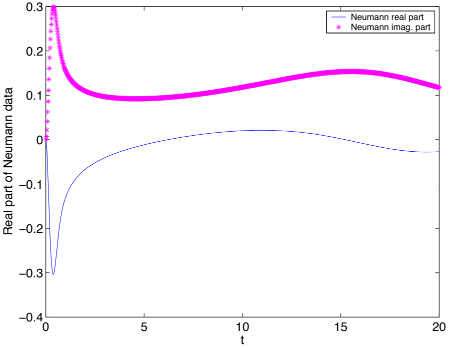
Numerics for $-3\alpha^2 < \omega < \alpha^2/2$, III

$\boxed{\alpha = 0.3, \quad \omega = 0}$

$q_0(x) \equiv 0, \;\; g_0(t) = \alpha + \mathrm{O}(\mathrm{e}^{-10t^2})$



Real part of $q(x,t)$

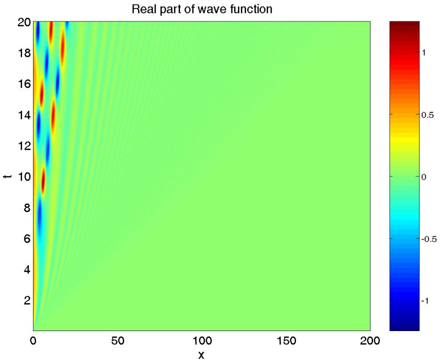


Neumann data

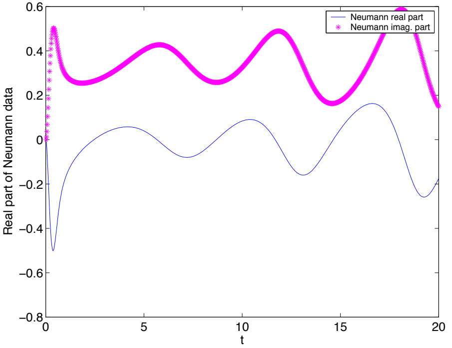
Numerics for $-3\alpha^2 < \omega < \alpha^2/2$, IV

$\boxed{\alpha = 0.5, \quad \omega = 0}$

$q_0(x) \equiv 0, \;\; g_0(t) = \alpha + \mathrm{O}(\mathrm{e}^{-10t^2})$



Real part of $q(x,t)$

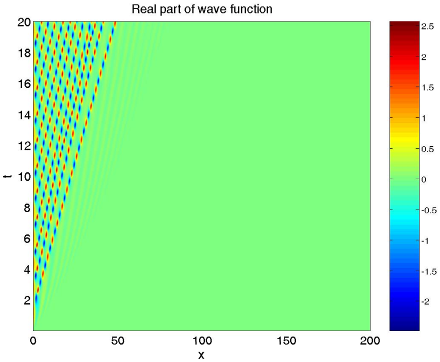


Neumann data

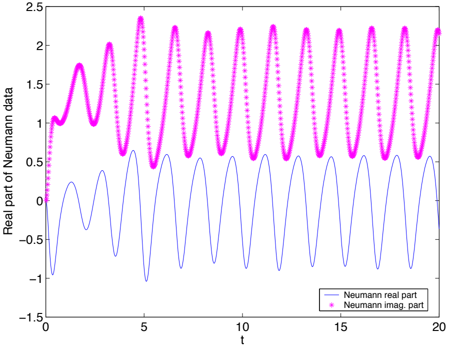
Numerics for $-3\alpha^2 < \omega < \alpha^2/2$, V

$\boxed{\alpha = 1, \quad \omega = 0}$

$q_0(x) \equiv 0, \; g_0(t) = \alpha + \mathrm{O}(\mathrm{e}^{-10t^2})$



Real part of $q(x,t)$



Neumann data

Linearizable cases: $q(0,t)=0$ or $q_x(0,t)=0$ or $q_x(0,t)+\rho q(0,t)=0$ (Robin b.c.)

(i) additional symmetry: $A(-k)=A(k)$, $B(-k)=-\frac{2k+\mathrm{i}\rho}{2k-\mathrm{i}\rho}B(k)$

(ii) global relation: $A(k)b(k)-B(k)a(k)=0$, $\mathrm{Im}\,k>0, \mathrm{Re}\,k>0$

(i)+(ii) allows "solving" $A(k)$, $B(k)$ in terms of $a(k)$, $b(k)$, so that the jump matrix for RHP can be expressed in terms of $a(k)$ and $b(k)$ (and ρ) only:

$$\tilde{C}(k)=\frac{\bar{b}(-\bar{k})}{a(k)}\,\frac{2k+\mathrm{i}\rho}{(2k-\mathrm{i}\rho)a(k)\bar{a}(-\bar{k})-(2k+\mathrm{i}\rho)b(k)\bar{b}(-\bar{k})}$$

Moreover, the RH problem on the cross can be deformed to RH problem on the real line (associated with initial value problem for NLS on the whole line)

Relationship to other problems

1. Novel integral representations for the solution of linear problems (A.S. Fokas: Unified Approach). For linear problems:
   (i) the Lax pair representation can be constructed algorithmically;
   (ii) the global relation that couples given initial and boundary data with unknown boundary values can be solved efficiently.
   - initial-boundary value problems for evolution PDEs containing x-derivatives of arbitrary order
   - elliptic equations in two variables (like the Laplace, the Helmholtz equations) formulated in the interior of a convex polygon
2. initial-value (Cauchy) problems with non-decaying (step-like) initial data

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